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On orthogonal and staircase connectedness in the plane

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Abstract

In this paper, we introduce *o*-extreme points defined by using orthogonal paths in orthogonally connected sets. We investigate their properties and obtain Minkowski-type theorems involving orthogonally connected sets. Using *o*-extreme points, we give some characterizations of staircase connectedness.

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1 INTRODUCTION

Convex sets are a very fruitful concept in geometry, having applications in optimization, statistics, computational geometry, functional analysis, and combinatorics. In several application areas of computational geometry, very large-scale integrated (VLSI) circuits and digital image processing, the only lines of interest are lines in the Euclidean plane parallel to the x- or y-coordinate axes. This is so, because orthogonal polygons, which are connected unions of finitely many planar boxes whose edges are parallel to the coordinate axes, are frequently used as building blocks for VLSI layout and wire routing, and also used in image processing to describe images on rectangular grids. This gives rise, in a natural way, to orthogonally convex sets.

Montuno and Fournier [12] introduced the notion of orthogonal convexity in orthogonal polygons. Nicholl, Lee, Liao, and Wong [13] also worked with the orthogonal convexity in orthogonal polygons, and gave the definition of a staircase. Ottmann, Soisalon-Soininen, and Wood [14] generalized the orthogonal convexity to any sets and defined orthogonal connectedness and staircase connectedness. Rawlins and Wood [15] gave some characterizations of orthogonally convex polygons.

A set $M \subset \mathbb{R}^2$ is *horizontally convex* (*vertically convex*), if M includes every horizontal (vertical) line-segment with endpoints in M. If M is both horizontally and vertically convex, we say that M is *orthogonally convex*. It is clear that convex sets are orthogonally convex.

A polygonal path P in $\mathbb{R}^2$ is called *orthogonal*, if every edge of P is parallel to one of the coordinate axes. A set $M \subset \mathbb{R}^2$ is *orthogonally connected*, if every two points $p, q \in M$ can be joined by an orthogonal path in M.

An orthogonal path is a *staircase*, if all horizontal edges point in the same direction, and all vertical edges point in the same direction. A set $M \subset \mathbb{R}^2$ is *staircase connected*, if every two points $p, q \in M$ can be joined by a staircase in M.

Breen investigated the staircase visibility in orthogonal polygons (see [1–4]). Magazanik and Perles [10] extended the investigation of staircase visibility from an orthogonal polygon to an arbitrary set.

Lemma 1.1 [10], Proposition 1.1. *A set in $\mathbb{R}^2$ is staircase connected if and only if it is orthogonally connected and orthogonally convex.*

For the staircase visibility, Breen established a Helly-type theorem for a finite family of simply connected orthogonal polygons [5], and then extended it to a family of planar compact sets having connected complements [6]. Moreover, she investigated the staircase connectedness of the union of a finite family $\mathcal{F}$ of boxes in $\mathbb{R}^d$, when the intersection graph of $\mathcal{F}$ is a tree [7], or a connected block graph [8].

For any convex set $M \subset \mathbb{R}^2$, a point of M is called an *extreme point* of M, if it is not the middle point of any line-segment in M. Let $\operatorname{ext}M$ denote the set of all extreme points of M.

Minkowski [11] proved the following fundamental theorem.

Theorem 1.2. *Let M be a compact convex subset of $\mathbb{R}^d$. Then $M = \operatorname{conv}(\operatorname{ext}M)$.*

We shall establish here theorems of Minkowski type.

The rest of this paper is organized as follows. We introduce o-extreme points defined by using orthogonal paths in orthogonally connected sets, and obtain Minkowski-type theorems in Section 2. Section 3 gives some characterizations of staircase connectedness using o-extreme points.

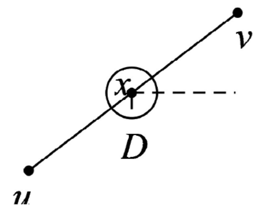

FIGURE 2.1 $x \in \text{oxt}M$.

The following notation will be used. For $a, b \in \mathbb{R}^2$, $\overline{ab}$ denotes the line through a and b. For any set $M \subset \mathbb{R}^2$, we denote by int$M$ the interior of M, by bd$M$ the boundary of M, by cl$M$ the closure of M, by $\complement M$ the complement of M, and by conv$M$ the convex hull of M. For $a_1, a_2, \ldots, a_n \in \mathbb{R}^2$, set

$$a_1 a_2 \cdots a_n = \text{conv}\{a_1, a_2, \ldots, a_n\}.$$

2 | *O*-EXTREME POINTS

Generalizing the classical notion of extremality from convexity, we say that, for any set $M \subset \mathbb{R}^2$, a point $x \in M$ is *extreme* in M, if it is not the middle point of any line-segment in M. The set of all extreme points in M is denoted by ext$M$.

Let $M \subset \mathbb{R}^2$ be orthogonally connected. A point $x \in M$ is *o-extreme* in M, if it belongs only as an endpoint to an orthogonal path in M. The set of all o-extreme points of M is denoted by oxt$M$.

Theorem 2.1. *If the topological disc M is orthogonally connected, then* $\text{oxt}M \subset \text{ext}M$.

Proof. Consider $x \in \text{oxt}M$ and suppose $x \notin \text{ext}M$. Then, there exist two points $u, v \in M$ such that $x \in \text{int}uv$ and $uv \subset M$. Since x is o-extreme, uv is neither horizontal nor vertical.

If $x \in \text{int}M$, then $x \notin \text{oxt}M$, whence $x \in \text{bd}M$, see Figure 2.1. Note that, since M is a topological disc, the boundary point x of M, which is the midpoint of the line-segment $uv \subset M$ is not the limit point of sequences of points not belonging to M and approaching from both sides of $\overline{uv}$. Indeed, if such sequences existed, the points could be joined by Jordan arcs in $\complement M$, and their limit would be a continuum C with $C \cap M = \{x\}$; thus, x would be a cut point of M (separating u from v), which is impossible. Take an open disc D of center x. If D is small enough, $D \setminus uv$ is disconnected, and one (and only one) component of $D \setminus uv$ is included in int$M$. Thus, in M there exist both horizontal and vertical line-segments starting from x, which means that x is not an o-extreme point of M, and a contradiction is obtained. □

Notice that, if M is not a topological disc, the inclusion does not necessarily hold. For example, let $M = abc \setminus \text{int}def$, where bc and de are horizontal, and $\angle edf > \frac{\pi}{2}$, see Figure 2.2. Then M is orthogonally connected and d is o-extreme, but not extreme.

In general, $\text{ext}M \not\subset \text{oxt}M$. For example, consider a rectangle with edges parallel to the coordinate axes. Each of its vertices is extreme, but not o-extreme. This example also shows that not every compact orthogonally connected set has o-extreme points. Clearly, for any orthogonally connected set M, if x is o-extreme, then $M \setminus \{x\}$ is orthogonally connected.

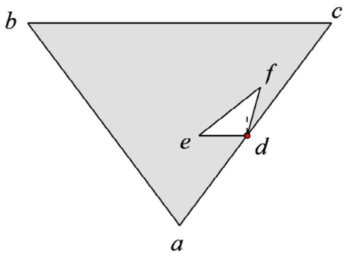

FIGURE 2.2 d is o-extremal, but not extremal.

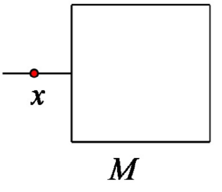

FIGURE 2.3 x is not o-extremal.

In [9], the following definitions appear. For $x \in M$, let

$$\mathrm{hw}_M(x) = \max_{y,y' \in M} \{\|y - y'\| : x \in yy' \subset M \text{ and } yy' \text{ is horizontal}\},$$

$$\mathrm{vw}_M(x) = \max_{y,y' \in M} \{\|y - y'\| : x \in yy' \subset M \text{ and } yy' \text{ is vertical}\}.$$

If M is orthogonally connected, for any $x \in M$,

$$\mathrm{hw}_M(x) + \mathrm{vw}_M(x) > 0,$$

but for any $x \in \mathrm{oxt}M$,

$$\mathrm{hw}_M(x) \cdot \mathrm{vw}_M(x) = 0.$$

However, this equality can hold without x being o-extreme (see the example in Figure 2.3).

We say that a line L *supports* a set $M \subset \mathbb{R}^2$ at a point $x \in M$, if $x \in L$ and M lies completely in one of the two closed half-planes determined by L. Next, we obtain a necessary and sufficient condition for a point to be an o-extreme point.

Theorem 2.2. *If the set M is staircase connected, then $x \in M$ is an o-extreme point if and only if there exists a horizontal or vertical supporting line L of M such that $L \cap M = \{x\}$.*

Proof. For the "only if" part, let $x \in \mathrm{oxt}M$. Then $x \in \mathrm{bd}M$. Since M is staircase connected, there exists a horizontal or vertical line-segment ux in M. Without loss of generality, suppose that ux is horizontal. Next we prove that there exists a vertical supporting line of M at x. Indeed, otherwise, there exists a point $v \in M$ such that u and v are separated by the vertical line L through x, see

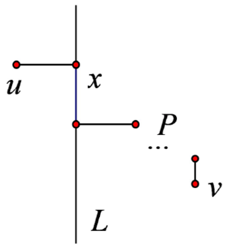

FIGURE 2.4 u and v are separated by the vertical line L.

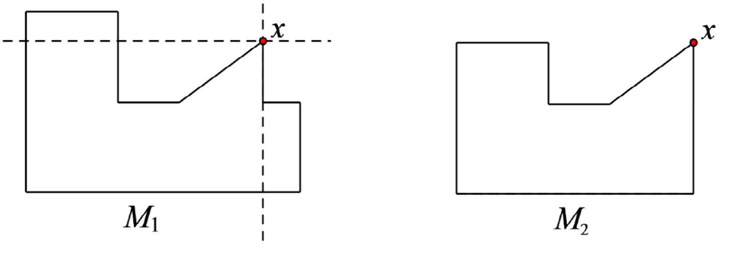

FIGURE 2.5 M_1 and M_2 are not staircase connected.

Figure 2.4. Due to the staircase connectedness of M, there exists a staircase $P \subset M$ that joins x and v. Hence, the staircase $ux \cup P \subset M$ contains x not as an endpoint, which contradicts $x \in \mathrm{oxt}M$. Thus, L is a supporting line of M. Because ux is horizontal and x is an o-extreme point, the staircase connectedness of M implies $L \cap M = \{x\}$.

For the "if" part, there exists a horizontal or vertical supporting line L of M at $x \in M$ such that $L \cap M = \{x\}$. Any staircase in M starting at x obviously has x as an endpoint. Therefore, $x \in \mathrm{oxt}M$. □

The staircase connectedness in Theorem 2.2 is necessary. For example, in Figure 2.5, both M_1 and M_2 are orthogonally connected, but not staircase connected. One can see that x is an o-extreme point of M_1, as well as M_2, but no horizontal or vertical line supports M_1 at x, or, if they do, both the horizontal and vertical lines through x intersect M_2 in more than one point.

For $A \subset M$, $\mathrm{oconv}_M(A)$ denotes the union of all orthogonal paths included in M with the endpoints in A. The following theorem and Corollary 2.5 give sufficient conditions for a set M to satisfy $\mathrm{oconv}_M(\mathrm{oxt}M) = M$.

Theorem 2.3. *If* $M \subset \mathbb{R}^2$ *is compact, orthogonally connected, and convex, and* $\mathrm{card}(\mathrm{oxt}M) \geqslant 2$, *then* $\mathrm{oconv}_M(\mathrm{oxt}M) = M$.

Proof. Since $\mathrm{card}(\mathrm{oxt}M) \geqslant 2$, we find two o-extreme points $u, v \in M$. There exist two line-segments, horizontal or vertical, $uu' \subset M$ and $vv' \subset M$.

Let $x \in M$. If $x \in \mathrm{oxt}M$, then obviously $x \in \mathrm{oconv}_M(\mathrm{oxt}M)$. If $x \notin \mathrm{oxt}M$, then there are two line-segments, horizontal or vertical, $xx_1 \subset M, xx_2 \subset M$, see Figure 2.6. We easily find two orthogonal paths P_u from u to x and P_v from v to x such that $P = P_u \cup P_v \subset uu'vv'x_2xx_1$ and $P_u \cap P_v = \{x\}$. Thus, x belongs to the path P, which joins u and v inside M, that is, $x \in \mathrm{oconv}_M(\mathrm{oxt}M)$. □

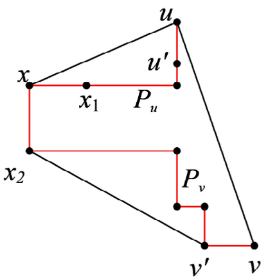

FIGURE 2.6 $x \notin \text{oxt}M$.

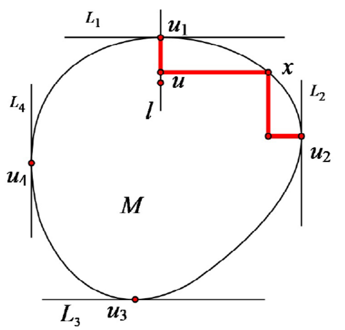

FIGURE 2.7 Horizontal lines L_1, L_3 support M.

Theorem 2.4. *If $M \subset \mathbb{R}^2$ is compact, orthogonally connected, and strictly convex, then* $\text{card}(\text{oxt}M) = 4$.

Proof. There are two distinct horizontal lines L_1, L_3 supporting M at u_1, u_3, respectively, see Figure 2.7. Since M is strictly convex, $L_1 \cap M = \{u_1\}, L_3 \cap M = \{u_3\}$, which implies that u_1 and u_3 are o-extreme. Analogously, the two vertical lines L_2, L_4 supporting M determine other points u_2, u_4. We prove that these four points u_1, u_2, u_3, u_4 are distinct. Indeed, suppose $u_1 = u_2$. We have $L_1 \cap M = \{u_1\}$ and $L_2 \cap M = \{u_1\}$. So, no horizontal or vertical line-segment included in M starts at u_1, contradicting the orthogonal connectedness of M.

Let x belong to $M \setminus \{u_1, u_2, u_3, u_4\}$. If $x \in \text{int}M$, then $x \notin \text{oxt}M$. If $x \in \text{bd}M$, without loss of generality suppose that x is between u_1 and u_2. There exists an orthogonal path passing through x, see Figure 2.7. Hence, x is not o-extreme. Therefore, only u_1, u_2, u_3, and u_4 are o-extreme points in M. □

We obtain the following corollary from Theorems 2.3 and 2.4.

Corollary 2.5. *If $M \subset \mathbb{R}^2$ is compact, orthogonally connected, and strictly convex, then* $\text{oconv}_M(\text{oxt}M) = M$.

For $A \subset M$, let $\text{sconv}_M(A)$ be the union of all staircases included in M with the endpoints in A. Clearly, $\text{sconv}_M(A) \subset \text{oconv}_M(A)$, but the inverse inclusion is not necessarily valid. For example,

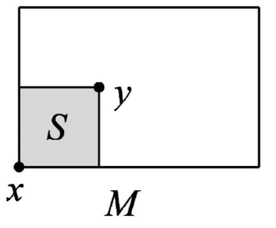

FIGURE 2.8 $\mathrm{oconv}_M(A) \not\subset \mathrm{sconv}_M(A)$.

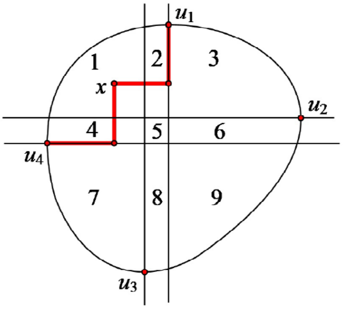

FIGURE 2.9 M.

let M be a rectangle, x be a vertex of M, and y be an interior point of M. Let S be the rectangle with xy as a diagonal, see Figure 2.8. Set $A = \{x, y\}$. Then $\mathrm{sconv}_M(A) = S$ and $\mathrm{oconv}_M(A) = M$.

The following result is a strengthening of Corollary 2.5.

Theorem 2.6. *If $M \subset \mathbb{R}^2$ is compact, orthogonally connected, and strictly convex, then* $\mathrm{sconv}_M(\mathrm{oxt}M) = M$.

Proof. From Theorem 2.4, we know that $\mathrm{card}(\mathrm{oxt}M) = 4$. Suppose $\mathrm{oxt}M = \{u_1, u_2, u_3, u_4\}$, see Figure 2.9. By definition, $\mathrm{sconv}_M(\mathrm{oxt}M) \subset M$. So we only need to show that M is a subset of $\mathrm{sconv}_M(\mathrm{oxt}M)$.

The set M can be divided into at most nine parts by the vertical lines through u_1, u_3 and the horizontal lines through u_2, u_4, see Figure 2.9. Assume $x \in M$ belongs to part 1. Figure 2.9 shows a staircase from u_4 to u_1 passing through x. This can be analogously done for any location of x. Thus, $M \subset \mathrm{sconv}_M(\mathrm{oxt}M)$. □

However, this does not remain true for M convex, but not strictly convex. For example, see Figure 2.10. In this case, $\mathrm{oxt}M = \{u_1, u_2, u_3\}$, $\mathrm{sconv}_M(\{u_1, u_2\}) = A \cup B$, $\mathrm{sconv}_M(\{u_2, u_3\}) = C$, $\mathrm{sconv}_M(\{u_1, u_3\}) = A \cup D$. Thus, $\mathrm{sconv}_M(\mathrm{oxt}M) = A \cup B \cup C \cup D \neq M$.

3 | ORTHOGONAL AND STAIRCASE CONNECTEDNESS

In [10], Magazanik and Perles proved that a set in the plane is staircase connected if and only if it is orthogonally connected and orthogonally convex. In this section, we shall look deeper into the relationship between orthogonal and staircase connectedness, and present some

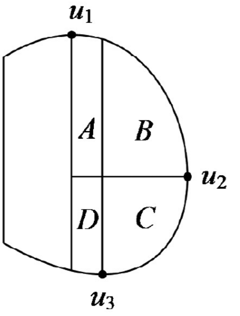

FIGURE 2.10 M is convex, but not strictly convex.

characterizations of staircase connected sets by using orthogonal connectedness and o-extreme points.

Theorem 3.1. *Any connected open set is orthogonally connected.*

Proof. For any connected open set A and $a \in A$, let

$$M_a = \{b \in A : \ a \text{ and } b \text{ are joined by an orthogonal path included in } A\},$$

and $S_a = A \setminus M_a$. It is clear that $M_a \neq \emptyset$. If $S_a \neq \emptyset$, it follows from the connectedness of A that $M_a \cap \mathrm{bd}S_a \neq \emptyset$ or $(\mathrm{bd}M_a) \cap S_a \neq \emptyset$.

If $M_a \cap \mathrm{bd}S_a \neq \emptyset$, let $b \in M_a \cap \mathrm{bd}S_a$. Since A is an open set, there exists an open ball $B \subset A$ with center b. We have $B \cap S_a \neq \emptyset$. Let $c \in B \cap S_a$. The points b and c are joined by an orthogonal path in B. Thus, a and c are connected by an orthogonal path in A, which contradicts $c \in S_a$.

If $(\mathrm{bd}M_a) \cap S_a \neq \emptyset$, let $b \in (\mathrm{bd}M_a) \cap S_a$. Since A is open, there exists an open ball $B \subset A$ with center b. Then $B \cap M_a \neq \emptyset$. Set $c \in B \cap M_a$. It is clear that the points b and c are connected by an orthogonal path in B. Thus, a and b are joined by an orthogonal path in A, which contradicts $b \in S_a$.

Therefore, $S_a = \emptyset$, so $M_a = A$. Due to the arbitrary choice of a, M is orthogonally connected. □

The following corollary immediately follows from Lemma 1.1 and Theorem 3.1.

Corollary 3.2. *Any connected open orthogonally convex set is staircase connected.*

A connected union of two orthogonally connected sets is obviously orthogonally connected. But it is not necessarily staircase connected, even if the two sets are both staircase connected. The same is true for convex bodies. For example, see Figure 3.2.

Theorem 3.3. *If the connected union M of finitely many smooth convex bodies is staircase connected, then* $\mathrm{cl}\complement M$ *is orthogonally connected and has no o-extreme point.*

Proof. For any $u \in \complement M$, there are horizontal line-segments and vertical line-segments containing u not as an endpoint and included in $\complement M$. For any $u \in \mathrm{bd}\complement M$, let L_1 and L_2 be the horizontal line

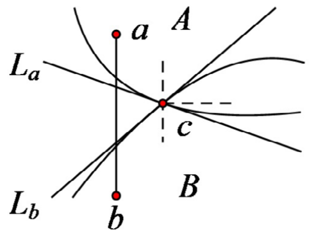

FIGURE 3.1 $ab \not\subset M$.

and the vertical line through u, respectively, and D be a small disc centered at u. Then D is divided into four parts by L_1 and L_2. Since M is staircase connected, there are at most three parts including points in M, which implies that there exists an orthogonal path through u included in $\mathrm{cl}\complement M$. So $\mathrm{cl}\complement M$ has no o-extreme point.

By Lemma 1.1, since M is staircase connected, M is orthogonally convex, and so it is simply connected. Hence, $\complement M$ is an unbounded and connected open set. Therefore, for any $u, v \in \mathrm{cl}\complement M$, there exist points $x, y \in \complement M$ such that the line-segments $ux, vy \subset \mathrm{cl}\complement M$ are horizontal or vertical. From Theorem 3.1, we know that $\complement M$ is orthogonally connected. Thus, u and v are connected by an orthogonal path in $\mathrm{cl}\complement M$. Hence, $\mathrm{cl}\complement M$ is orthogonally connected. □

Theorem 3.4. *The union of two smooth convex bodies having more than one point in common is staircase connected, if and only if the closure of its complement is orthogonally connected and has no o-extreme point.*

Proof. The "only if" part follows immediately from Theorem 3.3. So we only need to prove the "if" implication. We first show that, if A, B are the two convex bodies, $M = A \cup B$ is orthogonally convex. Suppose that M is not orthogonally convex. Then there exist two points $a \in A$ and $b \in B$ such that ab is vertical or horizontal, but $ab \not\subset M$. Without loss of generality, assume that ab is vertical. Let c be the closest point in $A \cap B$ to ab. Then c belongs to the relative boundary of $A \cap B$, whence it does not belong to the relative interior of $A \cap B$, and therefore $c \notin \mathrm{int}M$. Hence, $c \in \mathrm{bd}(\mathrm{cl}\complement M)$.

Consider the supporting lines L_a of A and L_b of B at the common point c. Since both L_a and L_b meet ab, see Figure 3.1, no vertical line-segment included in $\mathrm{cl}\complement M$ starts at c. Since $A \cap B$ is not a single point, no horizontal line-segment included in $\mathrm{cl}\complement M$ starts at c in the direction away from $\overline{ab}$. Hence, c is an o-extreme point of $\mathrm{cl}\complement M$, which contradicts the assumption that $\mathrm{cl}\complement M$ has no o-extreme points. Therefore, M is orthogonally convex.

Since A and B are smooth convex bodies, M is orthogonally connected. By Lemma 1.1, M is staircase connected. □

The condition $\mathrm{card}(A \cap B) \geqslant 2$ in Theorem 3.4 is necessary. For example, see Figure 3.2: The closure of the complement of $A \cup B$ has no o-extreme point, but $A \cup B$ is not staircase connected.

An extension of Theorem 3.4 follows.

Theorem 3.5. *Let $\mathcal{F}$ be a finite family of smooth convex bodies in $\mathbb{R}^2$ such that* $\mathrm{int} \cup \mathcal{F}$ *is connected and, for any triple $A, B, C \in \mathcal{F}$, there does not exist in the boundary of C any line-segment ab parallel to a coordinate axis such that $a \in A$ and $b \in B$. The set $\cup\mathcal{F}$ is staircase connected if and only if the closure of its complement is orthogonally connected and has no o-extreme point.*

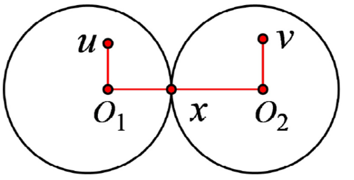

FIGURE 3.2 $A \cap B = \{x\}$.

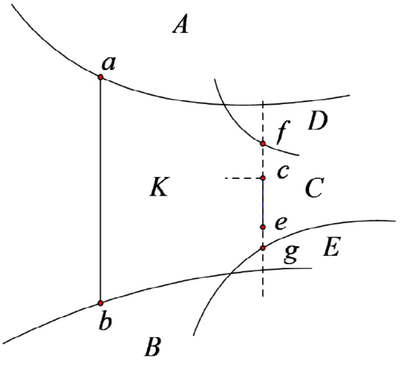

FIGURE 3.3 c is a farthest point of K from $\overline{ab}$.

Proof. The "only if" part follows directly from Theorem 3.3.

For the "if" part, we first show $M = \cup\mathcal{F}$ is orthogonally convex. Suppose that M is not orthogonally convex; then there exist two points $a, b \in M$ such that ab is vertical or horizontal, but $ab \not\subset M$. Without loss of generality, we assume that ab is vertical and minimal in the sense of inclusion, and $a \in A, b \in B$, where $A, B \in \mathcal{F}$. Let K be the closure of the bounded component of $\complement(M \cup ab)$ meeting ab and c be a farthest point of K from $\overline{ab}$ (see Figure 3.3). Then $c \in \mathrm{bd}K$. Clearly, $c \in \mathrm{bd}\complement M$. Since c is not an o-extreme point of $\mathrm{cl}\complement M$ and $\mathrm{int}M$ is connected, there exists a vertical line-segment $ce \subset \mathrm{bd}K$ such that $ce \subset C$ and $C \in \mathcal{F}$. Obviously, $ce \subset \mathrm{bd}C$. Since M is connected, there exist two point $f, g \in \overline{ce}$ and two convex bodies $D, E \in \mathcal{F}$ such that $f \in C \cap D$ and $g \in C \cap E$, which contradicts the hypothesis. Therefore, M is orthogonally convex.

Since M is a connected finite union of smooth convex bodies in $\mathbb{R}^2$, M is orthogonally connected. By Lemma 1.1, M is staircase connected. □

The following corollary is an immediate consequence of Theorem 3.5.

Corollary 3.6. *Let M be a finite union of strictly convex, smooth convex bodies in $\mathbb{R}^2$ with* $\mathrm{int}M$ *connected. The set M is staircase connected if and only if* $\mathrm{cl}\complement M$ *is orthogonally connected and has no o-extreme point.*

The condition about the triple A, B, C is necessary in Theorem 3.5. For example, let $M = A \cup B \cup C$ be like in Figure 3.4. One can see that $\mathrm{cl}\complement M$ is orthogonally connected and has no o-extreme points. However, C is not strictly convex and M is not staircase connected.

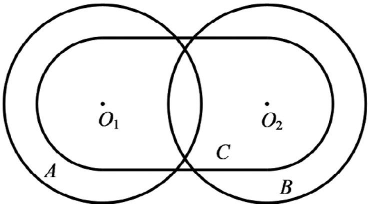

FIGURE 3.4 C is not strictly convex.

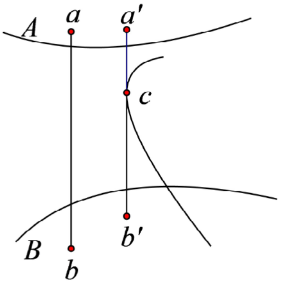

FIGURE 3.5 $ab \not\subset M$.

Theorem 3.7. *Let M be a connected union of finitely many strictly convex, smooth convex bodies. If no pair of convex bodies have precisely one point in common, and* $\mathrm{cl}\complement M$ *is orthogonally connected and has no o-extreme point, then M is staircase connected.*

Proof. We first show that $\mathrm{int}M$ is connected. For any two convex bodies A and B, if $A \cap B \neq \emptyset$, then $\mathrm{card}(A \cap B) \geqslant 2$, which implies that $\mathrm{int}(A \cup B)$ is connected.

A straightforward argument by induction on the number of convex bodies involved shows that $\mathrm{int}M$ is connected.

By Corollary 3.6, M is staircase connected. □

In the following, we further investigate the staircase connectedness of the union of a finite family of smooth convex bodies.

Theorem 3.8. *Let $\mathcal{F}$ be a finite family of smooth convex bodies, the union M of which is connected, such that, for any $A \in \mathcal{F}$,* $\mathrm{bd}A$ *includes no line-segment parallel to a coordinate axis. If, for any pair $A, B \in \mathcal{F}$ with connected union, $A \cup B$ is staircase connected, then M is staircase connected.*

Proof. We show that M is staircase connected by induction on $\mathrm{card}(\mathcal{F}) = n$.

When $n = 2$, the hypothesis directly implies the statement.

We assume that M is staircase connected for at most $n-1$ smooth convex bodies. Now we consider the case $\mathrm{card}(\mathcal{F}) = n \geqslant 3$. Suppose that M is not orthogonally convex; then there exist two points $a, b \in M$ such that ab is vertical or horizontal, but $ab \not\subset M$. Without loss of generality, we assume that ab is vertical, and $a \in A, b \in B$, where $A, B \in \mathcal{F}$, see Figure 3.5. Let G be the intersection graph of $\mathcal{F}$. The connectedness of $\mathcal{F}$ and that of G are equivalent. Let P be a shortest path in G connecting the vertices A and B. If P has at most $n-1$ vertices, then $ab \subset M$ by the induction hypothesis, contradicting our assumption. If P is longer, then its vertices form $\mathcal{F}$. Choose a

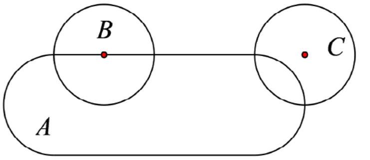

FIGURE 3.6 A is not strictly convex.

point $c \in \mathrm{cl}(M \setminus (A \cup B))$ closest to $\overline{ab}$. This point c belongs to some convex body $C \in \mathcal{F}$, vertex of P different from A and B. The subpaths of P from C to A and from C to B determine two connected unions of elements of $\mathcal{F}$ of cardinality less than n. There exist two points $a' \in A, b' \in B$ such that the line $\overline{a'b'} \ni c$ is parallel to ab. By the induction hypothesis, $a'c, cb' \subset M$. Since for any convex body of $\mathcal{F}$, there does not exist in its boundary any line-segment parallel to a coordinate axis, the same holds for M, which implies that c is not the closest point from $\overline{ab}$. Thus, we get a contradiction.

Therefore, M is orthogonally convex.

Since $\mathcal{F}$ is a family of smooth convex bodies, M is orthogonally connected.

It follows from Lemma 1.1 that M is staircase connected. □

The condition that there does not exist in the boundary of each smooth convex body any line-segment parallel to a coordinate axis in Theorem 3.8 is not redundant. For example, $A \cup B$ and $A \cup C$ shown in Figure 3.6 are staircase connected, but $A \cup B \cup C$ is not staircase connected.

From Theorem 3.8, we get the following result.

Corollary 3.9. *Let $\mathcal{F}$ be a finite family of strictly convex, smooth convex bodies, the union M of which is connected. If, for any pair $A, B \in \mathcal{F}$ with connected union, $A \cup B$ is staircase connected, then M is staircase connected.*

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